UNIT-2

(Lecture-4)

Design of Infinite Impulse Response Digital Filters: Bi-Linear Transformation

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Bilinear Transformation

The IIR filter design using approximation of derivatives method and the impulse invariant method are appropriate for the design of low-pass filters and band-pass filters whose resonant frequencies are low. These techniques are not suitable for high-pass or band-reject filters. This limitation is overcome in the mapping technique called the bilinear transformation. This transformation is a one-to-one mapping from the s-domain to z-domain. That is, the bilinear transformation is of conformal mapping that transforms the j Ω-axis into the unit circle in the z-plane only once thus avoiding aliasing of frequency components.

Bilinear Transformation

Also the transformation of a stable analog filter results in a stable digital filter as all the poles in the left half of the s-plane are mapped onto points inside the unit circle of z-domain.

The bilinear transformation is obtained by using the trapezoidal formula for numerical integration. Let the system function of the analog filter be

$$H(s) = \frac{b}{s + a} \quad \text{----------(1)}$$

The differential equation describe the analog filter can be obtain from equation(1) as

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$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$sY(s) + aY(s) = bX(s) \quad \text{----------(2)}$$

Taking inverse Laplace transform

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \quad \text{-----------(3)}$$

Now integrate the above equation between the limits (nT-T) and nT

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{nT} x(t)\,dt \quad \text{------(4)}$$

The trapezoidal rule for the numerical integration is given by -

Bilinear Transformation

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}\left[a(nT) + a(nT-T)\right] \qquad \text{-------(5)}$$

Applying equation(5) in equation (4) we get,

$$y(nT) - y(nT-T) + \frac{aT}{2} y(nT) + \frac{aT}{2} y(nT-T) = \frac{bT}{2} x(nT) + \frac{bT}{2} x(nT-T)$$

Taking z-transform , the system function of the digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a} \qquad \text{----------(6)}$$

Comparing equation(1) and (6) we get,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \qquad \text{----------(7)}$$

Bilinear Transformation

The general characteristics of the mapping $z=e^{sT}$ can be obtained by substituting $s=\sigma+j\Omega$ and expressing the complex variable z in the polar form as $z=re^{j\omega}$ in equation(7)

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

$$= \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) \text{------(8)}$$

Therefore,

$$\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right) \text{------------(9)}$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) \text{-----------(10)}$$

From eq. (9), it can be noted that if $r<1$, then $\sigma<0$, and if $r>1$, then $\sigma>0$. Thus the left half of the s-plane maps

Bilinear Transformation

Onto the points inside the unit circle in the z-plane and the transformation results in a stable digital system. Consider eq. (10), for unity magnitude(r=1), σ is zero. In this case,

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin\omega/2\cos\omega/2}{\cos^2\omega/2+\sin^2\omega/2+\cos^2\omega/2-\sin^2\omega/2}\right)$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

------(11)

Or equivalently, $\omega = 2\tan^{-1}\frac{\Omega T}{2}$ ------------(12)

Bilinear Transformation

Example: Applying Bilinear transformation to

$$H(s) = \frac{2}{(s+1)(s+3)}$$ **with T = 0.1s.**

Solution:

For bilinear transformation,

$$H(z) = H(s)\big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{2}{\left(\frac{2}{T}\frac{(z-1)}{(z+1)}+1\right)\left(\frac{2}{T}\frac{(z-1)}{(z+1)}+3\right)}$$

Using T = 0.1s,

$$H(z) = \frac{2}{\left(20\frac{(z-1)}{(z+1)}+1\right)\left(20\frac{(z-1)}{(z+1)}+3\right)}$$

$$= \frac{2(z+1)^2}{(21z-19)(23z-17)}$$

$$H(z) = \frac{0.0041\left(1+z^{-1}\right)^2}{1-1.644z^{-1}+0.668z^{-2}}$$

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